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Mathematics

Spinor representations of affine Lie algebras

(Clifford algebra/Clifford module/fundamental representation/quantum field theory/elliptic θ-functions)

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ABSTRACT Let $\mathfrak{g}$ be an infinite-dimensional Kac–Moody Lie algebra of one of the types $D_{l+1}^{(2)}$, $B_l^{(1)}$, or $D_l^{(1)}$. These algebras are characterized by the property that an elimination of any endpoint of their Dynkin diagrams gives diagrams of types B_l or D_l of classical orthogonal Lie algebras. We construct two representations of a Lie algebra $\mathfrak{g}$, which we call spinor representations, following the analogy with the classical case. We obtain that every spinor representation is either irreducible or has two irreducible components. This provides us with an explicit construction of fundamental representations of $\mathfrak{g}$, two for the type $D_{l+1}^{(2)}$, three for $B_l^{(1)}$, and four for $D_l^{(1)}$. We note the profound connection of our construction with quantum field theory—in particular, with fermion fields. Comparing the character formulas of our representations with another construction of the fundamental representations of Kac–Moody Lie algebras of types $A_l^{(1)}$, $D_l^{(1)}$, $E_l^{(1)}$, we obtain classical Jacobi identities and addition formulas for elliptic θ-functions.

1. Affine Orthogonal Lie Algebras. Let $\mathfrak{o}(n)$ denote a classical orthogonal Lie algebra over the complex field C. We consider a Cartan decomposition

$$\mathfrak{o}(2l+2) = \mathfrak{o}(2l+1) \oplus \mathfrak{e}, \mathfrak{e} \cong C^{2l+1}, l = 1, 2, \ldots \quad [1.1]$$

Let us fix a subalgebra $\mathfrak{o}(2l) \subset \mathfrak{o}(2l+1)$ and choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{o}(2l)$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{o}(2l+1)$ as well. The orthogonal algebras $\mathfrak{o}(2l+1)$ and $\mathfrak{o}(2l), l > 2$, are simple Lie algebras of types B_l and D_l, respectively (see ref. 1). Let $R(B_l)$ and $R(D_l)$ denote the root systems of pairs $(\mathfrak{o}(2l+1),\mathfrak{h})$ and $(\mathfrak{o}(2l),\mathfrak{h})$, and $S(B_l)$ and $S(D_l)$ be corresponding bases of root systems. Let $\langle , \rangle$ denote the bilinear symmetric invariant form on $\mathfrak{o}(2l+2)$ normalized in such a way that $\langle \alpha,\alpha \rangle = 2$, for $\alpha \in R(D_l)$. We will identify $\mathfrak{o}(2l+2)$ with its dual with respect to this form.

Let $C[t,t^{-1}]$ be the algebra of Laurent polynomials in the indeterminate t over C. We consider the following infinite-dimensional Lie algebra (see ref. 2)

$$\tilde{\mathfrak{g}} = \mathfrak{o}(2l+2) \otimes C[t,t^{-1}] \oplus Cc \oplus Cd \quad [1.2]$$

with Lie bracket defined by formulas

$$[X \otimes t^m, Y \otimes t^n] = [X,Y] \otimes t^{m+n} + \delta_{m,-n} m \langle X,Y \rangle c$$
$$[d, X \otimes t^m] = m \cdot X \otimes t^m, \quad [1.3]$$

in which $X,Y \in \mathfrak{o}(2l+2)$, $m,n \in Z$, and Cc is a one-dimensional center. We also define a bilinear symmetric form on $\tilde{\mathfrak{g}}$, setting

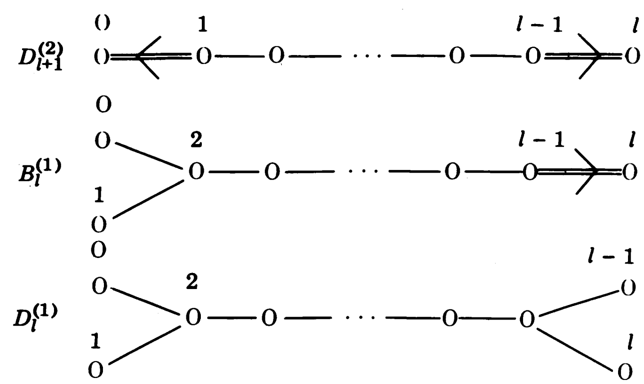
$$\langle X \otimes t^m, Y \otimes t^n \rangle = \delta_{m,-n} \langle X,Y \rangle$$
$$\langle c,d \rangle = 1, \langle c,c \rangle = \langle d,d \rangle = 0, \quad [1.4]$$

in which $X,Y \in \mathfrak{o}(2l+2)$, $m,n \in Z$ and $Cc \oplus Cd$ is orthogonal to $\mathfrak{o}(2l+2) \otimes C[t,t^{-1}]$. One can check (see ref. 2) that $\langle , \rangle$ is an invariant form, and we will identify $\tilde{\mathfrak{g}}$ with a linear subspace in its dual space with respect to this form.

Now we define three subalgebras in $\tilde{\mathfrak{g}}$ as follows:

$$\tilde{\mathfrak{o}}^{(2)}(2l+2) = \mathfrak{o}(2l+1) \otimes C[t^2,t^{-2}]$$
$$\times \oplus \mathfrak{e} \otimes t\,C[t^2,t^{-2}] \oplus Cc \oplus Cd$$
$$\tilde{\mathfrak{o}}(2l+1) = \mathfrak{o}(2l+1) \otimes C[t^2,t^{-2}] \oplus Cc \oplus Cd$$
$$\tilde{\mathfrak{o}}(2l) = \mathfrak{o}(2l) \otimes C[t^2,t^{-2}] \oplus Cc \oplus Cd. \quad [1.5]$$

We call these algebras affine orthogonal Lie algebras of rank $l+1$. According to the classification of ref. 3 they belong to the classes $D_{l+1}^{(2)}$, $B_l^{(1)}$, and $D_l^{(1)}$, respectively. We will identify $\mathfrak{h} \otimes 1$ with $\mathfrak{h}$ and we call $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus Cc \oplus Cd$ Cartan subalgebra of affine orthogonal Lie algebras in Eqs. **1.5.** Again we denote by $R(D_{l+1}^{(2)})$, $R(B_l^{(1)})$, and $R(D_l^{(1)})$ the corresponding root systems with respect to $\tilde{\mathfrak{h}}$, and we choose bases of root systems as follows: $S(D_{l+1}^{(2)}) = S(B_l) \cup \{c - \gamma_1\}$, γ_1 is a short dominant root of $R(B_l)$, $S(B_l^{(1)}) = S(B_l) \cup \{2c - \gamma_2\}$, γ_2 is a highest root of $R(B_l)$, $S(D_l^{(1)}) = S(D_l) \cup \{2c - \gamma_3\}$, γ_3 is a highest root of $R(D_l)$. The construction of the corresponding Dynkin diagrams is standard (see ref. 3).



[1.6]

The zero point of every Dynkin diagram corresponds to the additional element of the base—i.e., to $c - \gamma_1$, $2c - \gamma_2$, or $2c - \gamma_3$. Let $S = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ be a base of the root system of affine Lie algebra $\tilde{\mathfrak{o}}$. We denote by $\Omega = \{\omega_0, \omega_1, \ldots, \omega_l\}$ the fundamental weights, defined by the conditions $2\langle \omega_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle = \delta_{ij}$, $\langle \omega_i, d \rangle = 0$, $i,j = 0, 1, \ldots, l$. Let $\lambda = n_0\omega_0 + \ldots + n_l\omega_l$, in which $n_0, \ldots, n_l \in Z_+$; then, due to ref. 3, there exists exactly one irreducible highest-weight module V_λ of the affine Lie algebra $\tilde{\mathfrak{o}}$, with the highest weight λ. We will call $V_i = V_{\omega_i}$, $i = 0, 1, \ldots, l$, fundamental modules, and $V_0 = V_{\omega_0}$ the basic module.

2. Construction of Spinor Representations. We begin with the classical construction of spinor representations of orthogonal Lie algebras $\tilde{\mathfrak{o}}(n)$ (see, e.g., ref. 1). Let $C(n)$ denote a Clifford algebra of dimension 2^n over C; i.e., $C(n)$ is an associative al-

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gebra with unit 1 generated by the elements e_i, $i = 1, \ldots, n$, satisfying the relations

$$\{e_i,e_j\} = e_ie_j + e_je_i = -2\delta_{ij}, i,j = 1, 2, \ldots, n. \quad [2.1]$$

The Clifford algebra has a natural Z_2-grading $C(n) = C^0(n) \oplus C^1(n)$, in which $C^0(n)$ is a subalgebra of $C(n)$ spanned by the products of an even number of elements e_i, $i = 1, \ldots, n$. Let $n = 2l + 1$. We recall the construction of a simple $C(2l + 1)$ module, which we denote by $V(l)$. Let us introduce

$$b_r = \tfrac{1}{2}(ie_r + e_{r+l}), b_r^+ = \tfrac{1}{2}(ie_r - e_{r+l}), \quad r = 1, \ldots, l, \quad [2.2]$$

and let us denote by e the generator e_{2l+1}. Then by the definition

$$V(l) = \sum \mathcal{C}b_{r_1}^+ \ldots b_{r_k}^+ v_0, \quad [2.3]$$

in which the sum is taken by all the sets $1 \leq r_1 < \ldots < r_k \leq l$, including the empty set. The action of $C(2l + 1)$ can be defined from the conditions

$$b_rv_0 = 0, r = 1, \ldots, l, ev_0 = iv_0. \quad [2.4]$$

According to physical terminology we call $v_0 \in V(l)$ the vacuum vector and the action of the elements b_r^+, b_r, $r = 1, \ldots, l$, the creation and annihilation operators, respectively.

The module $V(l)$ has a natural Z_2-grading $V(l) = V^0(l) \oplus V^1(l)$ compatible with Z_2-grading of the Clifford algebra $C(2l + 1)$, $V^0(l)$ defined by Eq. **2.3** with even k, $V^1(l)$ with odd k.

Let us define now Lie bracket in $C(2l + 1)$ as usual

$$[x_1,x_2] = x_1x_2 - x_2x_1, x_1,x_2 \in C(2l + 1), \quad [2.5]$$

and let $\mathfrak{e}$ denote the vector space spanned by the e_i, $i = 1, \ldots, 2l + 1$. Then the vector space spanned by the elements x_1x_2, in which $x_1,x_2 \in \mathfrak{e}$ and $\{x_1,x_2\} = 0$, provided with the Lie bracket **2.5**, is isomorphic to the simple Lie algebra $\mathfrak{o}(2l + 1)$, and $\mathfrak{o}(2l + 1) \oplus \mathfrak{e}$ is isomorphic to $\mathfrak{o}(2l + 2)$. We fix a subalgebra $\mathfrak{o}(2l) \subset \mathfrak{o}(2l + 1)$ generated by products of the elements e_i, $i = 1, \ldots, 2l$. The $\mathfrak{o}(2l + 1)$ action in $V(l)$ is called a spinor representation and is irreducible. Its restriction to $\mathfrak{o}(2l)$ is decomposed into two irreducible components acting in $V^0(l)$ and $V^1(l)$ (see ref. 1).

Now we go over to the infinite-dimensional case, which has many common features with the classical case described above. Let $C(Z^n)$ denote an infinite-dimensional Clifford algebra generated by elements $e_i(m)$, $i = 1, \ldots, n$, $m \in Z$, in which Z denotes either $2Z$ or $2Z + 1$, satisfying the relations

$$\{e_i(k),e_j(m)\} = e_i(k)e_j(m) + e_j(m)e_i(k) = -2\delta_{ij}\delta_{k,-m}, i,j = 1, 2, \ldots n; k,m \in Z. \quad [2.6]$$

Let $n = 2l$. We define the simple $C(Z^{2l})$-module, which we denote by $V(Z^l)$, as follows. Let

$$b_r(m) = \tfrac{1}{2}(ie_r(m) + e_{r+l}(m)),$$
$$b_r^+(m) = \tfrac{1}{2}(ie_r(-m) - e_{r+l}(-m)), r = 1, \ldots, l, m \in Z. \quad [2.7]$$

Then for these elements the only nonzero anticommutators are

$$\{b_r(m),b_r^+(m)\} = 1, r = 1, \ldots, l, m \in Z. \quad [2.8]$$

We call $b_r^+(m)$, $m \geq 0$, $b_r(m)$, $m < 0$ creation operators and $b_r^+(m)$, $m < 0$, $b_r(m)$, $m \geq 0$ annihilation operators, in which $r = 1, \ldots, l$, $m \in Z$. We define $V(Z^l)$ to be the free module generated by the creation operators acting on the vacuum vector v_0. The action of $C(Z^{2l})$ is defined now by the condition

$$b_r^+(m)v_0 = 0, m < 0,$$
$$b_r(m)v_0 = 0, m \geq 0, r = 1, \ldots, l, m \in Z. \quad [2.9]$$

The module $V(Z^l)$, as in the classical case, has a natural Z_2-grading $V(Z^l) = V^0(Z^l) \oplus V^1(Z^l)$, defined in the same way. Also the Lie bracket in $C(Z^{2l})$ is defined as before (see Eq. **2.5**).

Let $x = \sum_{i=1}^{2l} a_ie_i$, $a_i \in \mathcal{C}$. We denote by $x(n)$ the element $\sum_{i=1}^{2l} a_ie_i(n)$. Let $X = x_1x_2 \in \mathfrak{o}(2l)$. Then we define

$$X(m) = \sum_{k \in Z} x_1(k)x_2(m - k), m \in 2Z. \quad [2.10]$$

This operator is well defined in $V(Z^l)$, because for every $v \in V(Z^l)$ only a finite number of terms in Eq. **2.10** does not annul v.

PROPOSITION 2.11. *Let* $X_1, X_2, X_3 \in \mathfrak{o}(2l)$ *and* $[X_1, X_2] = X_3$; *then* $[X_1(m), X_2(n)] = X_3(m + n) + m/2\, \delta_{m,-n} \langle X_1,X_2\rangle$, m, n $\in 2Z$.

Let us define now an operator D in $V(Z^l)$ by the formulas

$$Dv_0 = 0, [D,e_i(m)] = m\cdot e_i(m), m \in Z. \quad [2.12]$$

This implies immediately $[D, X(m)] = m\cdot X(m)$, $m \in 2Z$, $X \in \mathfrak{o}(2l)$. Therefore, we have obtained a representation of $\tilde{\mathfrak{o}}(2l)$ in the spaces $V(Z^l)$, $Z = 2Z$ or $2Z + 1$, in which $X \otimes t^m$ is represented by $X(m)$ [**2.10**], $X \in \mathfrak{o}(2l)$, d by D [**2.12**], and c by $id/2$. Following the analogy with the classical case we call these two representations spinor representations of $\tilde{\mathfrak{o}}(2l)$.

The space $V(Z^l)$ has been considered before but in physics rather than in mathematics. It appears in the quantum field theory as the space of states of l different fermions—e.g., quarks. Affine Lie algebras are also known in physics as current algebras with nontrivial Schwinger terms. Ten years ago two physicists, Bardakci and Halpern, in ref. 4 constructed a representation of the subalgebra $\tilde{\mathfrak{gl}}(l)$ of $\tilde{\mathfrak{o}}(2l)$ in the space $V((2Z + 1)^l)$ (see formulas 3.1–3.11 in ref. 4). At that time the theory of affine Lie algebras began to take its first steps.

We will now proceed to the spinor representations of the affine Lie algebra $\tilde{\mathfrak{o}}(2l + 1)$. In the classical case the spinor representations of $\mathfrak{o}(2l + 1)$ and $\mathfrak{o}(2l)$ are defined in the same space $V(l)$, but in the affine case it is no longer so. We have to extend our spaces $V(Z^l)$, described above.

Let $C(Z)$ be an infinite-dimensional Clifford algebra generated by elements $e(m)$, $m \in Z$, with relations **2.6** ($n = 1$). We call $e(-m)$, $m > 0$ creation operations and $e(m)$, $m > 0$ annihilation operators, $m \in Z$. As above we define $V(Z_+)$ and provide it with Z_2-grading. Now the Clifford algebra $C(Z^{2l+1}) \cong C(Z^{2l}) \otimes C(Z)$ is represented in a natural way in the space $\overline{V}(Z^l) = V(Z^l) \otimes V(Z_+)$, in which $\otimes$ denotes a Z_2-graded tensor product [$e(0)v_0 = iv_0$ as in **2.4**].

One can construct the representation of $\tilde{\mathfrak{o}}(2l + 1)$ in the spaces $\overline{V}(Z^l)$, $Z = 2Z$ or $2Z + 1$ if one repeats literally the construction of the spinor representation $\tilde{\mathfrak{o}}(2l)$, changing $2l$ to $2l + 1$ and $V(Z^l)$ to $\overline{V}(Z^l)$. We call these two representations spinor representations of $\tilde{\mathfrak{o}}(2l + 1)$.

Now we proceed to the third and last type of affine orthogonal Lie algebras: to the algebras $\tilde{\mathfrak{o}}^{(2)}(2l + 2)$. We again have to extend the $\tilde{\mathfrak{o}}(2l + 1)$-module $\overline{V}(Z^l)$ to obtain the representation of $\tilde{\mathfrak{o}}^{(2)}(2l + 2)$. First we consider one remarkable correspondence between irreducible representations of commutation and anticommutation relations.

Let $C(Z) = C(2Z) \otimes C(2Z + 1)$; i.e., $C(Z)$ is a Clifford algebra with generators $e(m)$, $m \in Z$, satisfying the relation

$$\{e(m),e(n)\} = -2\delta_{m,-n}, m,n \in Z. \quad [2.13]$$

And let $V(Z_+) = V(2Z_+) \otimes V((2Z + 1)_+)$ be a $C(Z)$-module. Then the following operators are well defined in $V(Z_+)$:

$$E(m) = \sum_{k \in Z} e(k)e(m - k), m \in 2Z + 1, \quad [2.14]$$

and one can check that

$$[E(m),E(n)] = -2m\delta_{m,-n}. \quad [2.15]$$

We will see in the next section that the representation of the Lie algebra generated by $E(m)$, $m \in 2Z+1$, is irreducible. Sometimes **2.15** and **2.13** are called commutation and anticommutation relations. Therefore we obtain their irreducible representations in the same space $V(Z_+)$!

We are now in a position to construct spinor representations of $\tilde{\mathfrak{o}}^{(2)}(2l+2)$. First we define the representation of the Clifford algebra $C(Z^{2l} \oplus Z) = C(Z^{2l}) \otimes C(Z)$ in the space $\overline{\overline{V}}(Z^l) = V(Z^l) \otimes V(Z_+)$ as usual. Then for $X = x_1x_2 \in \mathfrak{o}(2l+1)$ we define the operator $X(m)$, $m \in 2Z$, by formula **2.10**, and for $X = x \in \mathfrak{e}$ we define the operator $X(m)$, $m \in 2Z+1$, as follows

$$X(m) = \sum_{k\in Z} x(k)e(m-k), m \in 2Z+1, \qquad [\mathbf{2.16}]$$

in which $e(n)$, $n \in Z$, are generators of $C(Z)$. In particular, if $x = e \in \mathfrak{e}$, then $X(m) = E(m)$ is defined by **2.14**.

PROPOSITION 2.17. *Let* $X_1, X_2, X_3 \in \mathfrak{o}(2l+1)$ *or* $\mathfrak{e}$ *and* $[X_1,X_2] = X_3$; *then* $[X_1(m_1), X_2(m_2)] = X_3(m_1+m_2) + \frac{1}{2}m_1 \delta_{m_1,-m_2} \langle X_1,X_2\rangle$, $m_i \in 2Z$ *iff* $X_i \in \mathfrak{o}(2l+1)$, $m_i \in 2Z+1$ *iff* $X_i \in \mathfrak{e}$; $i = 1,2$.

We define now the operator D in $\overline{\overline{V}}(Z^l)$ as in **2.12**. This completes the construction of spinor representation of $\tilde{\mathfrak{o}}^{(2)}(2l+2)$ in the spaces $\overline{\overline{V}}(Z^l)$, $Z = 2Z$ or $2Z+1$. As above $X \otimes t^m$ is represented by $X(m)$, [**2.10, 2.16**], d by D, c by $id/2$. Our next task is to decompose the six spinor representation into irreducible components.

3. Main Theorem and Character Formulas. First we note that if the operator $e(0)$ is defined in a spinor representation, as we have in the cases $\overline{V}(Z^l)$, $\overline{V}((2Z)^l)$, then the Z_2-grading is not preserved by the action of the affine orthogonal Lie algebra. On the contrary, if $e(0)$ is not defined there, as we have in the cases $V(Z^l)$, $\overline{V}((2Z+1)^l)$, then the Z_2-grading is preserved. This follows immediately from the definitions **2.10** and **2.16** of the representations of these affine Lie algebras. We call the subrepresentations in the even and odd components of $V(Z^l)$ and $\overline{V}((2Z+1)^l)$ semispinor representations. The crucial point of our construction is the irreducibility of three spinor and six semispinor representations.

Here we will make note of two approaches. Detailed proofs will appear elsewhere. The first one is based on the fact that the associative algebra generated by spinor representation contains with the sums **2.10** and **2.16** all the particular summands as well, which follows from the Kac construction of the Casimir operator for any affine Lie subalgebra of $\tilde{\mathfrak{o}}$ (see ref. 3). The second approach is based on the definition of action of the group Z^l in the space of the spinor representation. We show that Z^l and the Heisenberg subalgebra (see ref. 2) generate the whole space of the spinor representation. Then the fact that $Z^l \cong Q(B_l)$ and $[Z^l:Q(D_l)] = 2$, in which Q is the root lattice, implies the result (the action of Q was defined in ref. 2).

One can calculate easily the highest weights of the irreducible components, and we obtain

THEOREM 3.1. (i) *Each of the spinor representations of* $\tilde{\mathfrak{o}}(2l)$ *in the spaces* $V((2Z)^l)$ *and* $V((2Z+1)^l)$ *is decomposed into two irreducible components according to its* Z_2*-grading. One has* $V^0((2Z+1)^l) \cong V_0(D_l^{(1)})$, $V^1((2Z+1)^l) \cong V_1(D_l^{(1)})$, $V^0((2Z)^l) \cong V_l(D_l^{(1)})$, $V^1((2Z)^l) \cong V_{l-1}(D_l^{(1)})$, *and* v_0, $x_{\omega_1}(0)v_0$, v_0, $x_{\omega_1}(0)v_0$ *are, respectively, highest-weight vectors.*

(ii) *The spinor representation of* $\tilde{\mathfrak{o}}(2l+1)$ *in the space* $\overline{V}(2Z+1)^l)$ *is decomposed into two irreducible components according to its* Z_2*-grading, and is irreducible in the space* $\overline{V}((2Z)^l)$. *One has* $\overline{V}^0((2Z+1)^l) \cong V_0(B_l^{(1)})$, $\overline{V}^1((Z+1)^l) \cong V_1(B_l^{(1)})$, $\overline{V}((2Z)^l) \cong V_l(B_l^{(1)})$, *and* v_0, $x_{\omega_1}(0)v_0$, v_0 *are, respectively, highest-weight vectors.*

(iii) *The spinor representations of* $\tilde{\mathfrak{o}}^{(2)}(2l+2)$ *in the spaces* $\overline{\overline{V}}((2Z+1)^l)$ *and* $\overline{\overline{V}}((2Z)^l)$ *are irreducible. One has* $\overline{\overline{V}}((2Z+1)^l) \cong V_0(D_{l+1}^{(2)})$, $\overline{\overline{V}}((2Z)^l) \cong V_l(D_{l+1}^{(2)})$. *In each case* v_0 *is a highest-weight vector.*

V_0, V_1, V_{l-1}, and V_l denote the fundamental modules according to numerations of the points of Dynkin diagrams given in **1.6**.

Let V be a $\tilde{\mathfrak{o}}$-module, and let $V = \sum_{n<N} V^n$ be a decomposition into the eigenspaces of operator d with the eigenvalue n. We will consider only such modules V that $V^n = \{0\}$, for sufficiently large n, and that $\dim V^n < +\infty$. Every V^n is decomposed into the direct sum of the subspaces $V^{n,\mu}$ with respect to $\mathfrak{h} \subset \mathfrak{o} \subset \tilde{\mathfrak{o}}$, $\mu \in P(D_l)$, in which P is the weight lattice. We define the character of V as a formal sum

$$\mathrm{ch}\, V = \sum_{n<N} \left(\sum_{\mu\in P} \dim V^{n,\mu} \cdot e^{\mu} \right) q^n, \qquad [\mathbf{3.2}]$$

in which $\sum_{\mu\in P} \dim V^{n,\mu} e^{\mu}$ is an element of the group algebra $\mathbb{C}(P)$, for every $n < N$. Let us calculate at first the character of $V(Z_+)$ in two ways, using the facts that $V(Z_+)$ is the irreducible representation of anticommutation **2.13** and also of commutation relations **2.15**,

$$\mathrm{ch}\, V(Z_+) = \prod_{n=1}^{\infty} (1+q^n) = \prod_{n=1}^{\infty} (1-q^{2n-1})^{-1}. \qquad [\mathbf{3.3}]$$

We have obtained the famous identity (see ref. 5). On the other hand, we can use this identity to prove the irreducibility of the representation of commutation relations. One can also calculate the characters of the spinor representations using the facts that

$$[h,b_r^+(m)] = \langle h,\varepsilon_r\rangle b_r^+(m)$$
$$[h,b_r(m)] = \langle h,-\varepsilon_r\rangle b_r(m), \qquad [\mathbf{3.4}]$$

in which $h \in \mathfrak{h}$, $r = 1, \ldots l$, and $\pm\varepsilon_r$ are short roots of $R(B_l)$. Therefore, we have

$$\mathrm{ch}\, V((2Z+1)^l) = \prod_{n=1}^{\infty} \prod_{r=1}^{l} (1+q^{2n-1}e^{\varepsilon_r})(1+q^{2n-1}e^{-\varepsilon_r}) \qquad [\mathbf{3.5}]$$

and similarly the character formulas of other spinor representations. By *Theorem 3.1* we obtain also the character formulas for the fundamental representations of affine orthogonal Lie algebras, corresponding to the endpoints of their Dynkin diagrams. On the other hand, if we know the character formulas of the fundamental representations we can derive the statements of the theorem. For example, it follows from the character formula for the basic representation of $\tilde{\mathfrak{o}}(2l)$ (see ref. 2) that

$$\mathrm{ch}\, V_0(D_l^{(1)}) + \mathrm{ch}\, V_1(D_l^{(1)}) = \prod_{n=1}^{\infty} (1-q^{2n})^{-l} \sum q^{\langle\gamma,\gamma\rangle} e^{\gamma}, \qquad [\mathbf{3.6}]$$

in which the sum is taken over all $\gamma = n_1\varepsilon_1 + \ldots + n_l\varepsilon_l$, $n_k \in Z$, $k = 1, \ldots, l$. The equality of the right sides of Eqs. **3.5** and **3.6** follows from the Jacobi formula (see below). It implies statement *i* of *Theorem 3.1*. The two other proofs of our theorem mentioned above do not use the character formulas for the fundamental representations.

At last we write down three more character formulas:

$$\mathrm{ch}\, \overline{\overline{V}}(2Z+1) = \prod_{n=1}^{\infty} (1+q^{2n-1}e^{\varepsilon})(1+q^{2n-1}e^{-\varepsilon})(1+q^n)$$
$$= \prod_{n=1}^{\infty} (1-q^n)^{-1} \sum_{n\in Z} e^{n\varepsilon} q^{n^2} (=\Theta_0(\varepsilon,q))$$

$$\mathrm{ch}\, \overline{\overline{V}}(2Z) = e^{\varepsilon/2} \prod_{n=1} (1+q^{2n-2}e^{\varepsilon})(1+q^{2n}e^{-\varepsilon})(1+q^n)$$

$$= \prod_{n=1}^{\infty} (1-q^n)^{-1} \sum_{n\in Z} e^{(n+1/2)\varepsilon} q^{n^2-n} (=\Theta_1(\varepsilon,q)). \quad [3.7]$$

These formulas follow from the isomorphism $\tilde{o}^{(2)}(4) \cong \tilde{o}(3)$ and formula of ref. 6 for basic representation of $\tilde{o}(3)$.

$$\begin{aligned} \mathrm{ch}\, V((2Z+1)^2) &= c(q)\cdot \Theta_0(\varepsilon_1,q)\Theta_0(\varepsilon_2,q) \\ &= \Theta_0(\varepsilon_1-\varepsilon_2,q^2)\Theta_0(\varepsilon_1+\varepsilon_2,q^2) \\ &\quad + q\cdot\Theta_1(\varepsilon_1-\varepsilon_2,q^2)\Theta_1(\varepsilon_1+\varepsilon_2,q^2) \quad [3.8] \end{aligned}$$

in which $c(q) = \Pi_{n=1}^{\infty}(1+q^n)^{-2}$. The last formula follows from *Theorem 3.1* i for $l = 2$, and the fact that the character of a fundamental representation of $\tilde{o}(4)$ is equal to the product of characters of fundamental representations of $\tilde{o}(3)$.

The formulas of 3.7 are famous Jacobi formulas for θ-functions (see, e.g., ref. 5) and the formula of 3.8 is an addition formula in the theory of θ-function (see, e.g., ref. 5). Thus these classical formulas of the last century have found their geometrical interpretation in the isomorphism of two different realizations of the infinite-dimensional representations of affine Lie algebras.*

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* After this work was done and discussed at the Lie groups seminar at Yale University, the author, in conversation with V. G. Kac, learned that V. G. Kac and D. H. Peterson were working in a similar direction and have obtained some results.

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